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WAVE PROPAGATION IN A LAYERED ELASTIC PLATE

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Abstract—Formal solutions to a broad class of transient, symmetric, elastic wave problems involving a three layered plate are presented in terms of the modes of harmonic wave propagation. An approximation to the lowest mode is developed and, using this and the formal solutions, far-field expressions are developed for three specific problems. These approximate expressions are analyzed numerically for two specific cases.

INTRODUCTION

In a recent work [1], the author treated certain aspects of transient elastic wave propagation in an infinite inhomogeneous plate. There the material properties were taken to vary continuously in the thickness direction. The current paper extends that work to situations where discrete layering occurs.

Elastic transients in layered media have attracted attention over the years due to their importance in the fields of seismology and laminated composites. In seismology, considerable work has been done on the case of a single layer over a half space (see Newlands [2], Pekeris *et al.* [3, 4], Mitra [5], Laster *et al.* [6]). In the case of two layers over a half space, the work of Mitra [7] on SH-transients should be noted. The studies on composites are somewhat different in that most authors confine their attention to infinite periodic media, i.e. to a two-layered structure that repeats itself indefinitely. Voelker and Achenbach [8] presented approximate and exact results for transients generated by a time-dependent body force in such a medium. Peck and Gurtman [9] studied transients in a half space, the material of which was periodic. They gave and analyzed a long wave approximation, valid for the far-field (the head of the pulse approximation). In the same area, the work of Sve [10] using the effective stiffness theory of Achenbach *et al.* [11] should be noted.

Here, formal solutions are given to a broad class of symmetric, plane-strain, transient, wave propagation problems involving a three layered plate. The solution technique, which hinges on the modes of harmonic wave propagation, is similar to that used in [1] and is related to the methods used by the author [12], Rosenfeld and Miklowitz [13], Achenbach [14, 15], Reismann [16], Lee and Reismann [17], Wu and Plunkett [18] and Cobble [19]. The items necessary for the head of the pulse approximation are developed and then the approximation is applied to three specific problems, namely the longitudinal impact and mixed pressure shock problems for a semi-infinite plate and the sudden, symmetric application of normal line loads to an infinite plate.

MODES OF HARMONIC WAVE PROPAGATION

As in [1], solutions to transient problems will be sought in terms of the modes of harmonic wave propagation for an infinite plate. As opposed to [1] though, now the scheme is one of convenience rather than necessity in that, in principle, the modes can be found analytically. However, the current method leads to head of the pulse approximations more rapidly.†

The plate geometry and the coordinates used are shown in Fig. 1. This geometry is equivalent, for the inputs used here, to two perfectly bonded elastic layers of thickness h_1 and $h_2 - h_1$, respectively, in lubricated contact with a rigid half-space. In the sequel, the various problems will be examined in this latter context. The stress equations of motion are, in the absence of body forces, and assuming plane-strain conditions,

$$\frac{\partial_j\sigma_{xx}}{\partial x}+\frac{\partial_j\sigma_{xz}}{\partial z} = \rho_j\frac{\partial^2{}_ju_x}{\partial t^2}, \qquad j = 1, 2, \tag{1}$$

$$\frac{\partial_j\sigma_{xz}}{\partial x}+\frac{\partial_j\sigma_{zz}}{\partial z} = \rho_j\frac{\partial^2{}_ju_z}{\partial t^2}, \qquad j = 1, 2, \tag{2}$$

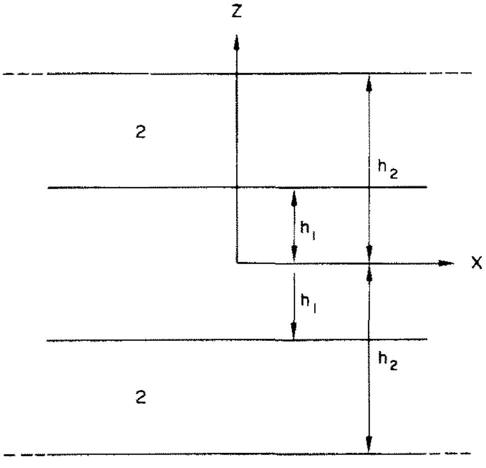

FIG. 1. Geometry of the layered plate.

where ρ denotes material density, σ stress, u displacement and the subscript j the layer in question. The stress–displacement relations are

$${}_j\sigma_{xx} = \lambda_j\left(\frac{\partial_ju_x}{\partial x}+\frac{\partial_ju_z}{\partial z}\right)+2\mu_j\frac{\partial_ju_x}{\partial x}, \qquad j = 1, 2, \tag{3}$$

$${}_j\sigma_{zz} = \lambda_j\left(\frac{\partial_ju_x}{\partial x}+\frac{\partial_ju_z}{\partial z}\right)+2\mu_j\frac{\partial_ju_z}{\partial z}, \qquad j = 1, 2, \tag{4}$$

$${}_j\sigma_{xz} = \mu_j\left(\frac{\partial_ju_x}{\partial z}+\frac{\partial_ju_z}{\partial x}\right), \qquad j = 1, 2, \tag{5}$$

† Presumably the same is true for problems such as that of Peck and Gurtman [9].

where λ and μ are Lamé's constants. The boundary conditions are

$$_2\sigma_{zz} = 0, \qquad _2\sigma_{xz} = 0, \qquad z = h_2 \tag{6}$$

$$_1\sigma_{xz} = 0, \qquad _1u_z = 0, \qquad z = 0. \tag{7}$$

Also, the assumption of a perfect bond requires the displacements and stresses to be continuous at $z = h_1$.

To obtain the modes of harmonic wave propagation, one sets

$$_ju_x(x, z, t) = {_ju_x^N}(z)\, \mathrm{e}^{ikx+p_Nt}, \qquad j = 1, 2, \tag{8}$$

$$_ju_z(x, z, t) = {_ju_z^N}(z)\, \mathrm{e}^{ikx+p_Nt}, \qquad j = 1, 2, \tag{9}$$

where k denotes wave number, p_N a frequency like term and the index N stands for the fact that an infinite number of modes exist. Substituting (8) and (9) into (3)–(5), yields modal stress–displacement relations. Typical ones are

$$_j\sigma_{xx}^N = ik(\lambda_j+2\mu_j)\,{_ju_x^N}+\lambda_j\frac{\mathrm{d}_ju_z^N}{\mathrm{d}z}, \qquad j = 1, 2. \tag{10}$$

Substituting the modal stress–displacement relations into (1) and (2), gives the modal equations of motion, typical ones being

$$\frac{\mathrm{d}_j\sigma_{xz}^N}{\mathrm{d}z}+ik\,{_j\sigma_{xx}^N} = \rho_jp_N^2\,{_ju_x^N}, \qquad j = 1, 2. \tag{11}$$

From (6) and (7), the modal boundary conditions are

$$_2\sigma_{zz}^N = 0, \qquad _2\sigma_{xz}^N = 0, \qquad z = h_2 \tag{12}$$

$$_1\sigma_{xz}^N = 0, \qquad _1u_z^N = 0, \qquad z = 0. \tag{13}$$

Also, the modal displacements and stresses must be continuous at $z = h_1$.

To develop pertinent properties of the modes the following Hermitian bilinear forms are introduced:

$$T(u^N, u^M) = 2\rho_1\int_0^{h_1}({_1u_z^N}\,{_1u_z^{M*}}+{_1u_x^N}\,{_1u_x^{M*}})\,\mathrm{d}z+2\rho_2\int_{h_1}^{h_2} \times({_2u_z^N}\,{_2u_z^{M*}}+{_2u_x^N}\,{_2u_x^{M*}})\,\mathrm{d}z \tag{14}$$

$$V(u^N, u^M) = 2\int_0^{h_1}\left[\frac{\mathrm{d}_1u_z^N}{\mathrm{d}z}\,{_1\sigma_{zz}^{M*}}+\frac{\mathrm{d}_1u_x^N}{\mathrm{d}z}\,{_1\sigma_{xz}^{M*}}+ik \times({_1u_x^N}\,{_1\sigma_{xx}^{M*}}+{_1u_z^N}\,{_1\sigma_{xz}^{M*}})\right]\mathrm{d}z+2\int_{h_1}^{h_2}\left[\frac{\mathrm{d}_2u_z^N}{\mathrm{d}z}\,{_2\sigma_{zz}^{M*}} +\frac{\mathrm{d}_2u_x^N}{\mathrm{d}z}\,{_2\sigma_{xz}^{M*}}+ik({_2u_x^N}\,{_2\sigma_{xx}^{M*}}+{_2u_z^N}\,{_2\sigma_{xz}^{M*}})\right]\mathrm{d}z \tag{15}$$

where * denotes complex conjugate. Using integration by parts, equations such as (10)–(13) and the continuity conditions at $z = h_1$, it can be shown that

$$V(u^N, u^M) = -p_M^{2*}T(u^N, u^M). \tag{16}$$

Using (16) and the Hermitian properties and taking complex conjugates, it can be shown that, on defining

$$T(u^N, u^N) = 1 \tag{17}$$

then

$$T(u^N, u^M) = \delta_{NM} \tag{18}$$

where δ_{NM} denotes the Kronecker delta.

FORMAL SOLUTIONS TO TRANSIENT PROBLEMS

Attention will now be focused on a class of transient problems. Taking the Laplace transform and the Fourier transform defined by†

$$\tilde{f}(k) = \int_0^\infty f(x)\,\mathrm{e}^{-ikx}\,\mathrm{d}x = \tilde{f}^C - i\tilde{f}^S \tag{19}$$

where C and S denote cosine and sine transforms, respectively, of (1) and (2), gives the transformed equations of motion. Typical ones are

$$\frac{\mathrm{d}_j\tilde{\bar{\sigma}}_{xz}}{\mathrm{d}z} + ik\,_j\tilde{\bar{\sigma}}_{xx} - {}_j\bar{\sigma}_{xx}\Big|_{x=0} = \rho_j p^2\,_j\tilde{\bar{u}}_x, \qquad j = 1, 2, \tag{20}$$

where the bar denotes the Laplace transform, parameter p. Taking Laplace and Fourier transforms of (3)–(5) yields the transformed stress–displacement relations, typical ones being

$$_j\tilde{\bar{\sigma}}_{xx} = \lambda_j \frac{\mathrm{d}_j\tilde{\bar{u}}_z}{\mathrm{d}z} + ik(\lambda_j + 2\mu_j)\,_j\tilde{\bar{u}}_x - (\lambda_j + 2\mu_j)\,_j\bar{u}_x\Big|_{x=0}, \qquad j = 1, 2. \tag{21}$$

The quadruplets of modal displacements $({}_1u_x^N, {}_2u_x^N, {}_1u_z^N, {}_2u_z^N)$, $N = 1, 2, \ldots$, can be interpreted as the characteristic vectors in a vector space consisting of quadruplets of functions. Then, assuming completeness, arbitrary quadruplets can be expressed in terms of the modal displacements. In particular, the quadruplet of transformed displacements arising in transient problems can be expanded in this fashion. One obtains

$$_j\tilde{\bar{u}}_x = \sum_N \Gamma_N\,_ju_x^N, \qquad j = 1, 2, \tag{22}$$

$$_j\tilde{\bar{u}}_z = \sum_N \Gamma_N\,_ju_z^N, \qquad j = 1, 2, \tag{23}$$

where the Γ_N are expansion coefficients.

It can be shown, using (14), (17), (18), (22) and (23), that

$$\Gamma_N = T(\tilde{\bar{u}}, u^N). \tag{24}$$

Using the bilinear form $V(\tilde{\bar{u}}, u^N)$, integration by parts, equations such as (20) and (21), the continuity conditions at $z = h_1$ and the modal boundary conditions, it can be shown that

$$V(\tilde{\bar{u}}, u^M) = -p_M^{2*}T(\tilde{\bar{u}}, u^M) \tag{25}$$

† Later, by considering only the odd or even parts of the functions of k that arise, this definition allows one to handle the longitudinal impact and pressure shock problems simultaneously.

where $V(\tilde{u}, u^M)$ is given by (15), with u^N replaced by $\tilde{u}$. To proceed further, it must be noted that

$$V(u^N, \tilde{u}) = 2\int_0^{h_1}\left[\frac{\mathrm{d}_1u_z^N}{\mathrm{d}z}({}_1\tilde{\bar{\sigma}}_{zz}^*)_v + \frac{\mathrm{d}_1u_x^N}{\mathrm{d}z}({}_1\tilde{\bar{\sigma}}_{xz}^*)_v + ik_1u_x^N({}_1\tilde{\bar{\sigma}}_{xx}^*)_v + ik_1u_z^N({}_1\tilde{\bar{\sigma}}_{xz}^*)_v\right]\mathrm{d}z$$
$$+2\int_{h_1}^{h_2}\left[\frac{\mathrm{d}_2u_z^N}{\mathrm{d}z}({}_2\tilde{\bar{\sigma}}_{zz}^*)_v + \frac{\mathrm{d}_2u_x^N}{\mathrm{d}z}({}_2\tilde{\bar{\sigma}}_{xz}^*)_v + ik_2u_x^N({}_2\tilde{\bar{\sigma}}_{xx}^*)_v + ik_2u_z^N({}_2\tilde{\bar{\sigma}}_{xz}^*)_v\right]\mathrm{d}z \tag{26}$$

where the subscript v denotes the fact that the transformed item in question is for an infinite plate (the geometry for which the properties of T and V were developed), in contrast to the actual transformed items for a semi-infinite plate, denoted in the sequel by a subscript P. The two are related, typical relationships being

$$({}_j\tilde{\bar{\sigma}}_{zz})_v = ({}_j\tilde{\bar{\sigma}}_{zz})_P + \lambda_j\,{}_j\bar{u}_x|_{x=0}, \qquad j = 1, 2. \tag{27}$$

Using equations such as (20), (21) and (27), integration by parts and the continuity conditions at $z = h_1$, it can be shown that (26) leads to

$$V(u^N, \tilde{u}) = B_{hN} + \bar{B}_{ON} - p^{2*}T(u^N, \tilde{u}) \tag{28}$$

where

$$B_{hN} = 2[{}_2u_z^N({}_2\tilde{\bar{\sigma}}_{zz}^*)_P + {}_2u_x^N({}_2\tilde{\bar{\sigma}}_{xz}^*)_P]|_{z=h_2} \tag{29}$$

$$\bar{B}_{ON} = 2\int_0^{h_1}[{}_1\sigma_{xx}^N\,{}_1\bar{u}_x^*|_{x=0} + {}_1\sigma_{xz}^N\,{}_1\bar{u}_z^*|_{x=0} - {}_1u_z^N\,{}_1\bar{\sigma}_{xz}^*|_{x=0} - {}_1u_x^N\,{}_1\bar{\sigma}_{xx}^*|_{x=0}]\,\mathrm{d}z + 2\int_{h_1}^{h_2}[{}_2\sigma_{xx}^N\,{}_2\bar{u}_x^*|_{x=0} + {}_2\sigma_{xz}^N\,{}_2\bar{u}_z^*|_{x=0} - {}_2u_z^N\,{}_2\bar{\sigma}_{xz}^*|_{x=0} - {}_2u_x^N\,{}_2\bar{\sigma}_{xx}^*|_{x=0}]\,\mathrm{d}z. \tag{30}$$

For problems involving an infinite plate, B_{ON} is set equal to zero. Using (24), (25) and (28), and the Hermitian properties of the bilinear forms, it can be shown that

$$(p^2 - p_N^{2*})\Gamma_N = B_{hN}^* + \bar{B}_{ON}^*. \tag{31}$$

Equations (22), (23), (29)–(31), constitute the formal solutions to a broad class of transient, symmetric, elastic wave propagation problems. Such solutions are very convenient starting points for approximate studies, in that the modes of harmonic wave propagation do not have to be known exactly. The particular approximation to be pursued here, which is accurate in the far-field, is the head of the pulse approximation.

FAR-FIELD APPROXIMATION

The dominant, early arriving, far-field disturbance is described, for symmetric waves in homogeneous, infinite rods and plates, by the low-frequency, large wave length portion of the lowest mode. The same is taken to be true for the case at hand. To obtain the required mode approximation, the following wave number expansions are employed:

$$p_1^2 = P_0 + (ik)^2 P_1 + (ik)^4 P_2 + \dots \tag{32}$$

$$_j u_x^1 = {}_j u_{x0} + (ik)^2\, {}_j u_{x1} + (ik)^4\, {}_j u_{x2} + \dots, \qquad j = 1, 2, \tag{33}$$

$$_j u_z^1 = (ik)\, {}_j u_{z1} + (ik)^3\, {}_j u_{z2} + (ik)^5\, {}_j u_{z3} + \dots, \qquad j = 1, 2, \tag{34}$$

where P_0, P_1, P_2, etc., are constants and ${}_j u_{x0}$–${}_j u_{z3}$ are at most functions of z.

Substituting (32)–(34) into equations such as (10)–(13) and (17), and the continuity conditions at $z = h_1$, sets of relations, which are power series in k, are obtained. On selectively terminating these series, groups of relations for the determination of the unknowns in (32)–(34) are found. To order k^0, one gets

$$\frac{\mathrm{d}\,{}_j\sigma_{xz1}}{\mathrm{d}z} = \rho_j P_0\, {}_j u_{x0}, \qquad j = 1, 2,$$

$${}_j\sigma_{xz1} = \mu_j \frac{\mathrm{d}\,{}_j u_{x0}}{\mathrm{d}z}, \qquad j = 1, 2,$$

$${}_2\sigma_{xz1} = 0, \qquad z = h_2$$

$${}_1\sigma_{xz1} = {}_2\sigma_{xz1}, \qquad z = h_1$$

$${}_1 u_{x0} = {}_2 u_{x0}, \qquad z = h_1$$

$${}_1\sigma_{xz1} = 0, \qquad z = 0$$

$$\rho_1 \int_0^{h_1} {}_1 u_{x0}^2 \,\mathrm{d}z + \rho_2 \int_{h_1}^{h_2} {}_2 u_{x0}^2 \,\mathrm{d}z = 1/2.$$

These equations are satisfied by

$$P_0 = 0 \tag{35}$$

$${}_1 u_{x0} = {}_2 u_{x0} = 1/\sqrt{\{2[\rho_1 h_1 + \rho_2 (h_2 - h_1)]\}}. \tag{36}$$

Proceeding as in [1], it can be shown that the relations to order k and k^2, which must be considered together to get a determinate system, can be satisfied by

$${}_1 u_{z1} = az, \qquad {}_2 u_{z1} = bz + l \tag{37}$$

$${}_1 u_{x1} = cz^2 + mz + d, \qquad {}_2 u_{x1} = fz^2 + qz + g \tag{38}$$

by a suitable choice of the constants a–g. The process is somewhat lengthy, and, in the interests of brevity, the values of the constants will not be given here. It can also be shown that the equations to order k^3 and k^4 (the terminal stage for the head of the pulse approximation) can be satisfied by†

$${}_1 u_{z2} = Az^3 + Bz, \qquad {}_2 u_{z2} = \Phi z^3 + Qz^2 + Dz + F$$

$${}_1 u_{x2} = Gz^4 + Sz^3 + Iz^2 + T_1 z + J$$

$${}_2 u_{x2} = Kz^4 + V_1 z^3 + Lz^2 + Wz + M$$

by an appropriate choice of the constants A–M.

† This is a departure from the method used in [1], where at this stage the equations of motion were used. The current technique is felt to be more straightforward and more capable of generalization.

As regards the lowest mode approximation, one finds, after considerable algebra, that

$$\rho_2 P_1 = \mu_2 \Gamma \tag{39}$$

$$\rho_2 P_2 = \mu_1 h_1^2 \alpha \tag{40}$$

where†

$$\Gamma = \frac{2R_\rho}{R_\rho(1-R_h)-1}\left[\frac{1-R_h}{1-\sigma_2} - \frac{1}{R_\mu(1-\sigma_1)}\right]$$

$$R_\rho = \rho_2/\rho_1, \qquad R_\mu = \mu_2/\mu_1, \qquad R_h = h_2/h_1$$

$$\alpha = \frac{R_\rho\Delta_1 - R_\mu\Delta_2}{1+R_\rho(R_h-1)} - \frac{R_\mu\Psi_2}{1-2\sigma_2} + \left[\frac{2(1-\sigma_2)}{1-2\sigma_2} - \Gamma\right]R_\mu\Psi_1$$

$$\Delta_1 = R_\mu(1-R_h^3)(4\zeta-\beta) + R_\mu(1-R_h^2)(3\xi+\Psi) - R_\mu(1-R_h)\Psi_2 - 4\eta + \gamma_3 - \gamma_1$$

$$\Delta_2 = -\frac{\Psi_2}{1-2\sigma_2} + \left[\frac{2(1-\sigma_2)}{1-2\sigma_2} - \Gamma\right]\Psi_1 - \frac{R_\rho}{R_\mu(1-2\sigma_1)}\gamma_1$$
$$-\left[\frac{2R_\rho(1-\sigma_1)}{R_\mu(1-2\sigma_1)} - \Gamma\right](\Theta - \gamma + \Delta + \Psi_1)$$

$$\Psi_2 = R_h^2\left[\frac{\sigma_2\Theta}{1-\sigma_2} - 3\beta\right] + R_h\left[2\Psi + \frac{\sigma_2\Delta}{1-\sigma_2}\right] + \frac{\sigma_2\Psi_1}{1-\sigma_2}$$

$$\Psi_1 = \left\{\frac{\sigma_1^2}{3(1-\sigma_1)^2} - \tfrac{2}{3}R_\rho\Theta(R_h^3-1) - \tfrac{2}{3}\gamma - [R_\rho(R_h^2-1)+2]\Delta\right.$$
$$+\frac{\sigma_2^2 R_\rho(R_h^3-1)}{3(1-\sigma_2)^2} + \frac{R_\rho(R_h-1)(\sigma_2-\sigma_1)^2}{(1-\sigma_2)^2(1-\sigma_1)^2}$$
$$\left.-\frac{R_\rho\sigma_2(R_h^2-1)(\sigma_2-\sigma_1)}{(1-\sigma_1)(1-\sigma_2)^2} - 2(\Theta-\gamma)\right\}[2+2R\rho(Rh-1)]^{-1}$$

$$\zeta = \left[\Gamma - \frac{2(1-\sigma_2)}{1-2\sigma_2}\right]\frac{\Theta}{12} + \frac{\beta}{4(1-2\sigma_2)}$$

$$\beta = \frac{\sigma_2(1-2\sigma_2)}{12(1-\sigma_2)^2}(\Gamma-1) + \frac{\Theta}{6(1-\sigma_2)}$$

$$\xi = \left[\Gamma - \frac{2(1-\sigma_2)}{1-2\sigma_2}\right]\frac{\Delta}{6} - \frac{\Psi}{3(1-2\sigma_2)}$$

$$\Psi = \frac{(1-2\sigma_2)(\sigma_2-\sigma_1)(\Gamma-1)}{4(1-\sigma_1)(1-\sigma_2)^2} - \frac{\Delta}{4(1-\sigma_2)}$$

$$\eta = \left[\frac{R_\mu\Gamma}{R_\rho} - \frac{2(1-\sigma_1)}{1-2\sigma_1}\right]\frac{\gamma}{12} + \frac{\gamma_3}{4(1-2\sigma_1)}$$

$$\gamma_3 = \frac{\sigma_1(1-2\sigma_1)}{12(1-\sigma_1)^2}\left(\frac{R_\mu\Gamma}{R_\rho} - 1\right) + \frac{\gamma}{6(1-\sigma_1)}$$

† Though the ensuing expressions are quite complicated, no effort has been made to simplify them further, since very little would be gained by this from the view point of digital computation.

$$\gamma_1 = 3\gamma_3 + \left[\frac{R_\mu \sigma_2(1-2\sigma_1)}{(1-\sigma_1)(1-2\sigma_2)} - \frac{\sigma_1}{1-\sigma_1}\right](\Theta + \Delta + \Psi_1)$$

$$+\frac{R_\mu(1-\sigma_2)(1-2\sigma_1)}{(1-\sigma_1)(1-2\sigma_2)}(2\Psi - 3\beta - \Psi_2)$$

$$2\Theta = \frac{1}{R_\rho(1-R_h)-1}\left[\frac{2-\sigma_2+\sigma_2 R_\rho(1-R_h)}{1-\sigma_2} - \frac{2R_\rho}{R_\mu(1-\sigma_1)}\right]$$

$$2\gamma = \frac{1}{R_\rho(1-R_h)-1}\left[\frac{2R_\mu(1-R_h)}{1-\sigma_2} + \frac{R_\rho(1-R_h)(\sigma_1-2)-\sigma_1}{1-\sigma_1}\right]$$

$$\Delta = \frac{2R_h}{R_\rho(1-R_h)-1}\left[\frac{R_\rho}{R_\mu(1-\sigma_1)} - \frac{1}{1-\sigma_2}\right] - \frac{\sigma_2-\sigma_1}{(1-\sigma_1)(1-\sigma_2)}$$

where σ_1 and σ_2 denote Poisson's ratio for materials one and two, respectively.

Equations (35)–(40) are sufficient to implement the head of the pulse approximation, which is now applied to the following three basic problems: (i) Longitudinal impact problem. The edge boundary conditions are

$$_1u_x = {}_2u_x = u_0 S(t), \qquad x = 0 \tag{41}$$

$$_1\sigma_{xz} = {}_2\sigma_{xz} = 0, \qquad x = 0 \tag{42}$$

where u_0 is a constant and $S(t)$ denotes the Heaviside unit step function. Also, the plate is taken to be stress free at $z = h_2$. (ii) Mixed pressure shock problem. The edge boundary conditions are

$$_1u_z = {}_2u_z = 0, \qquad x = 0 \tag{43}$$

$$_1\sigma_{xx} = {}_2\sigma_{xx} = -\sigma_0 S(t), \qquad x = 0 \tag{44}$$

where σ_0 is a constant. Again, the plate is taken to be stress free at $z = h_2$. (iii) Line load problem. The boundary conditions at $z = h_2$ are taken as

$$_2\sigma_{zz} = -L_0 S(t)\delta(x), \qquad z = h_2 \tag{45}$$

$$_2\sigma_{xz} = 0, \qquad z = h_2 \tag{46}$$

where L_0 is a constant and δ denotes the delta function

Inserting (41)–(46) into (29), (30) and (31), the boundary terms can be evaluated. Using (19) and retaining only the odd function of k in these terms gives the transformed solution (Laplace and Fourier sine or cosine) to the longitudinal impact problem, whereas the transformed solutions (Laplace and Fourier sine or cosine) to the pressure shock problem stem from the even functions of k. The full expressions are retained for the line load problem, but B_{ON} is set equal to zero. In all cases the Laplace transforms can be inverted by residue theory and the Fourier transforms can be inverted by means of the appropriate inversion theorems. Then, applying the head of the pulse approximation (see [20] for details), some typical results for the three problems are:

(i)
$$
{}_1u_x = {}_2u_x = u_0\left(\frac{1}{3}+\int_0^{\psi'} Ai(-\eta)\,d\eta\right) \tag{47}
$$

(ii)
$$
\frac{\mu_1\,{}_1\varepsilon_{zz}}{\sigma_0} = \frac{\sigma_1 R_h}{2\left[1-R_\mu(1-R_h)\left(\dfrac{1-\sigma_1}{1-\sigma_2}\right)\right]}\left[\frac{1}{3}+\int_0^{\psi'} Ai(-\eta)\,d\eta\right] \tag{48}
$$

$$
\frac{\mu_1\,{}_2\varepsilon_{zz}}{\sigma_0} = \frac{\sigma_2(1-\sigma_1)R_h}{2(1-\sigma_2)\left[1-R_\mu(1-R_h)\left(\dfrac{1-\sigma_1}{1-\sigma_2}\right)\right]}\left[\frac{1}{3}+\int_0^{\psi'} Ai(-\eta)\,d\eta\right] \tag{49}
$$

$$
\frac{\mu_1\,{}_1\varepsilon_{xx}}{\sigma_0} = \frac{\mu_1\,{}_2\varepsilon_{xx}}{\sigma_0} = -\frac{(1-\sigma_1)R_h}{2\left[1-R_\mu(1-R_h)\left(\dfrac{1-\sigma_1}{1-\sigma_2}\right)\right]}\left[\frac{1}{3}+\int_0^{\psi'} Ai(-\eta)\,d\eta\right] \tag{50}
$$

(iii)
$$
\frac{\mu_1\,{}_1u_x}{L_0} = \frac{\mu_1\,{}_2u_x}{L_0} = \frac{(1-\sigma_1)\left[\dfrac{\sigma_2(1-R_h)}{1-\sigma_2}-\dfrac{\sigma_1}{1-\sigma_1}\right]}{4\left[1-R_\mu(1-R_h)\left(\dfrac{1-\sigma_1}{1-\sigma_2}\right)\right]}\left[\frac{1}{3}+\int_0^{\psi'} Ai(-\eta)\,d\eta\right] \tag{51}
$$

where Ai denotes the Airy function, ε strain and

$$
\psi' = \frac{1}{\tau^{\frac{1}{3}}}\left(\frac{2\sqrt{(R_\rho R_\mu \Gamma)}}{3\alpha}\right)^{\frac{1}{3}}\left(-\zeta_1+\tau\sqrt{\frac{R_\mu \Gamma}{R_\rho}}\right)
$$

$$
\tau = \frac{t}{h_1}\sqrt{\left(\frac{\mu_1}{\rho_1}\right)}, \qquad \zeta_1 = x/h_1.
$$

NUMERICAL RESULTS

Some general observations should first be made, namely: (a) The horizontal displacements in materials one and two are identical in the longitudinal impact problem. Moreover, the amplitude of the pulse is independent of material properties and the thickness of the layers. However, the periods that arise do depend on these items. (b) The horizontal strains in materials one and two in the pressure shock problem are identical. (c) The horizontal displacements in materials one and two in the line load problem are identical.

Shown in Figs. 2–7 are the results of some sample computations of (47)–(51). Figures 2–6 exhibit the effects of the thickness ratio R_h on the various responses at the station $\zeta_1 = 80$. As regards the other parameters, two cases are treated, namely, $\sigma_1 = \sigma_2 = 0{\cdot}25$, $R_\mu = 13{\cdot}792275$, $R_\rho = 1{\cdot}39$ (case I), which is an example of seismological interest (see Grant and West [21, p. 84]) and $\sigma_1 = 0{\cdot}20$, $\sigma_2 = 0{\cdot}41$, $R_\mu = 0{\cdot}00834711$, $R_\rho = 0{\cdot}451711$ (case II), which is an example treated by Peck and Gurtman [9] and corresponds to layer one being boron and layer two being epoxy. Figure 2 gives the horizontal displacements in case I for the longitudinal impact problem. It is seen that the greater R_h the earlier the arrival of the main disturbance. However, R_h appears to have very little effect on the periods. Figures 3 and 4 show the horizontal strains in the pressure shock problem for cases I and II, respectively. It is seen that the amplitudes and periods decrease with R_h. For case II, the

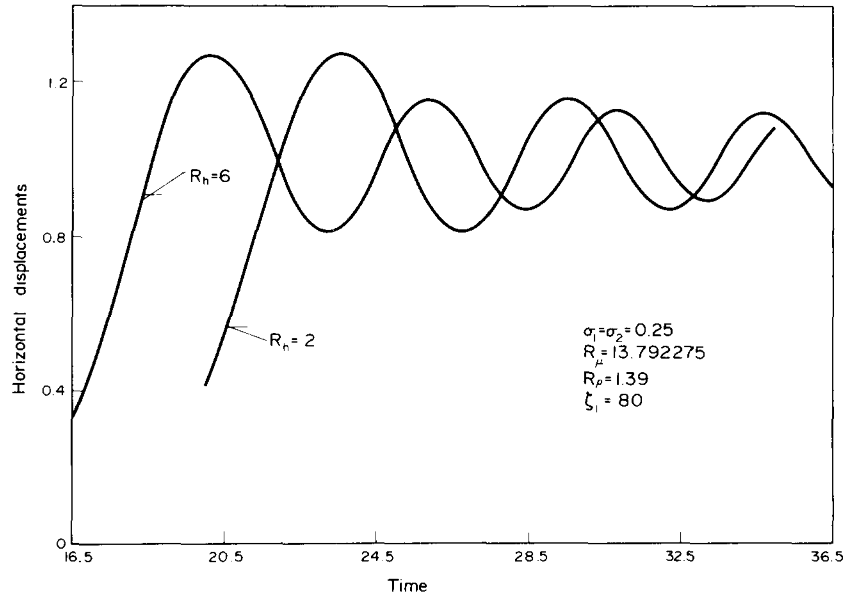

FIG. 2. Horizontal displacements u_x/u_0 vs. time $\tau = t/h_1\sqrt{(\mu_1/\rho_1)}$ in the longitudinal impact problem. Case I.

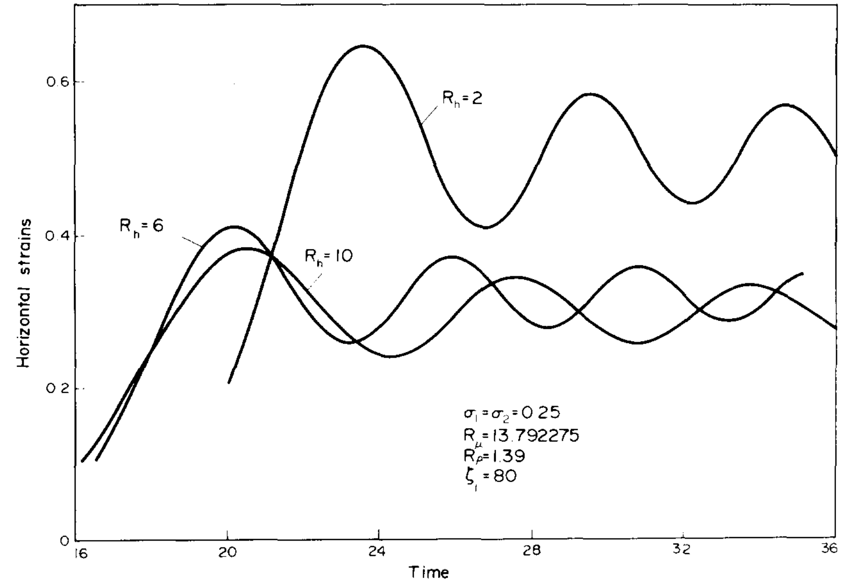

FIG. 3. Horizontal strains $-\mu_1\varepsilon_{xx}/\sigma_0$ vs. time $\tau = t/h_1\sqrt{(\mu_1/\rho_1)}$ in the pressure shock problem. Case I.

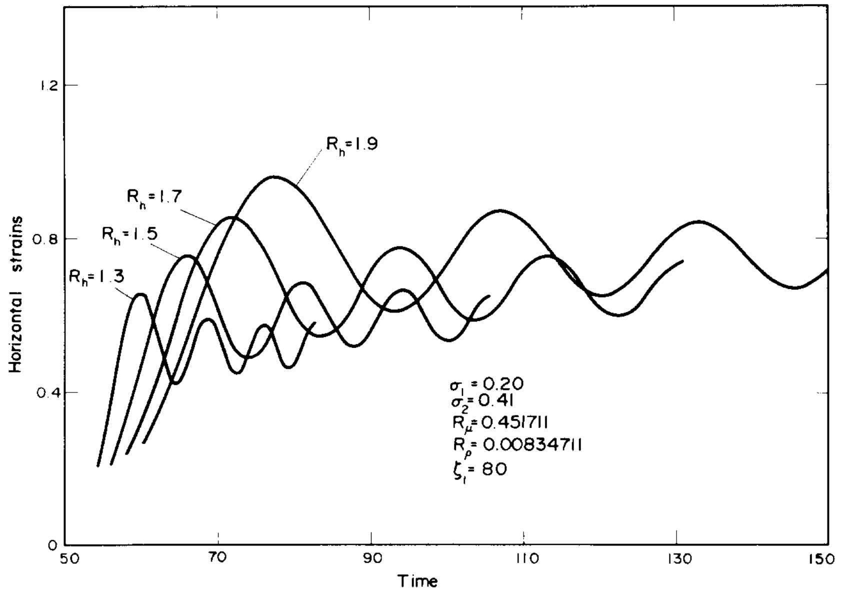

FIG. 4. Horizontal strains $-\mu_1\varepsilon_{xx}/\sigma_0$ in the pressure shock problem versus time $\tau = t/h_1\sqrt{(\mu_1/\rho_1)}$. Case II.

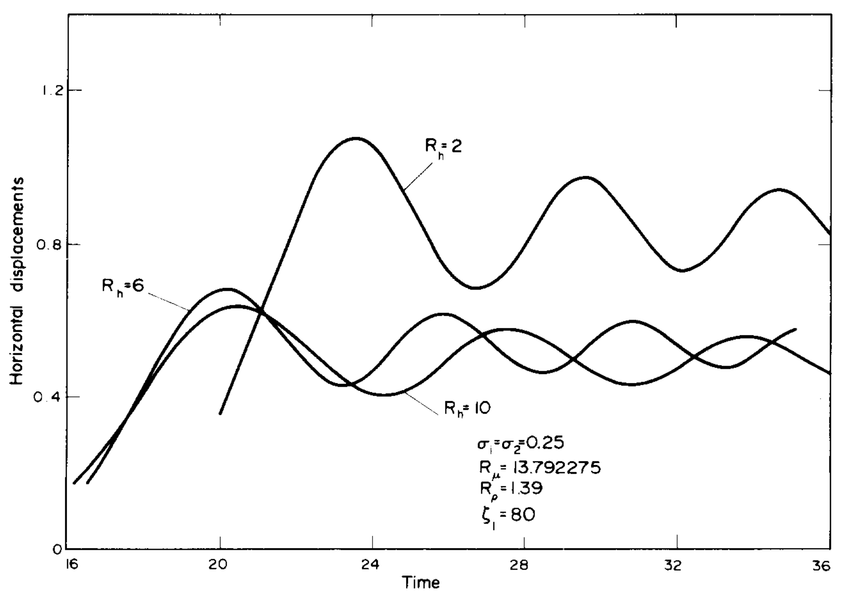

FIG. 5. Horizontal displacements $-\mu_1\mu_x/L_0$ vs. time $\tau = t/h_1\sqrt{(\mu_1/\rho_1)}$ in the line load problem. Case I.

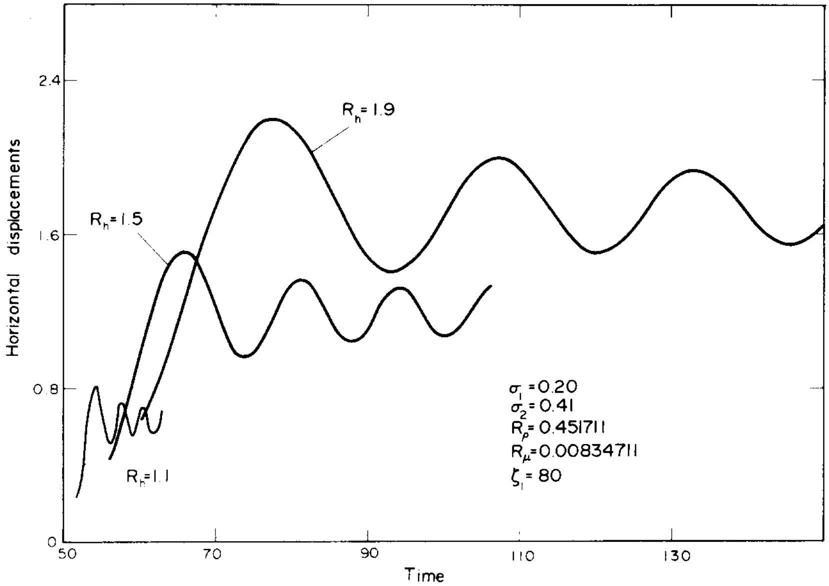

Fig. 6. Horizontal displacements $-\mu_1 u_x/u_0$ vs. time $\tau = t_1/h_1\sqrt{(\mu_1/\rho_1)}$. Case II.

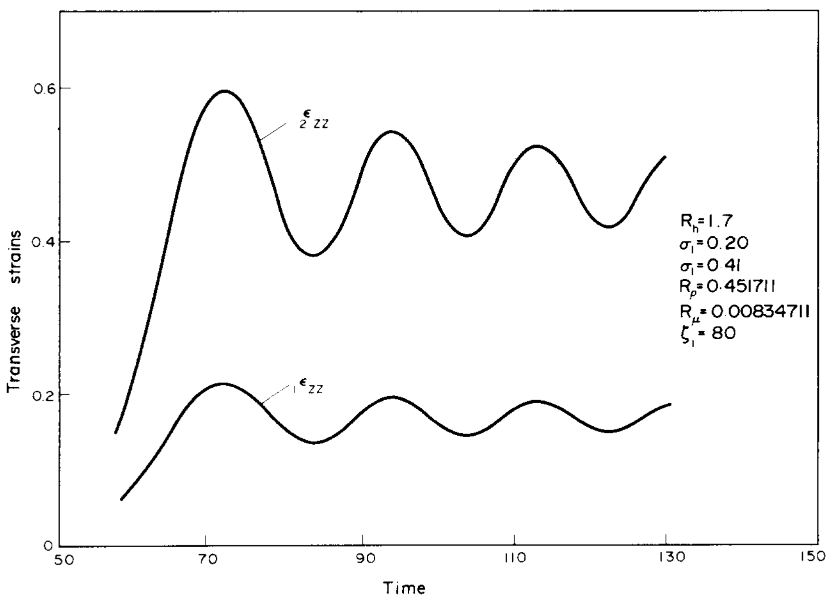

Fig. 7. Transverse strains $\mu_{11}\varepsilon_{zz}/\sigma_0$ and $\mu_{12}\varepsilon_{zz}/\sigma_0$ vs. time $\tau = t/h_1\sqrt{(\mu_1/\rho_1)}$ in the pressure shock problem. Case II.

situation is reversed. The larger R_h, the greater are the amplitudes and periods. Figures 5 and 6 give the horizontal displacements in the line load problem for cases I and II, respectively. The main features that emerge are that in case I the greater R_h, the smaller the amplitudes but the larger the periods, whereas for case II, both amplitudes and periods increase with R_h.

Shown in Fig. 7 are the transverse strains for case II in the pressure shock problem at the station $\zeta_1 = 80$. It is seen that the strains in layer two are considerably larger than those in layer one.

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Абстракт—Представляются формальные решения, выраженные зависимостями распространения гармонической волны, для широкого класса нестационарных, симметрических, упругих волн в трехслойных пластинках. Дается приближение самого малого вида колебаний. Используя этот факт и формальные решения, даются выражения для удаленных полей трех спеуифических задач. Обсуждаются численно этие приближенные выражения для двух специфических случаев.